20^{th} BALKAN MATHEMATICAL OLYMPIAD
2003

1. Is there a set of 4004 positive integers so that the sum of any 2003 of them be not divisible by 2003?

2. Let ABC be a triangle with $AB \neq AC$, and let D be the point of intersection between the tangent at A to the circumcircle of ABC and BC. Consider the points E, F which lie on the perpendiculars raised from B and C to BC, and on the perpendicular bisectors of AB and AC, respectively. Prove that D,E and F are collinear.

3. Find all functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ that satisfy the conditions:

   (a) $f(1) + 1 > 0$.

   (b) $f(x + y) - xf(y) - yf(x) = f(x)f(y) - x - y + xy$, for all $x, y \in \mathbb{Q}$.

   (c) $f(x) = 2f(x + 1) + x + 2$, for every $x \in \mathbb{Q}$.

4. Let $ABCD$ be a rectangle of side lengths m, n made out of $m \times n$ unit squares. Assume that m and n are two odd and coprime positive integers. The points of intersection between the main diagonal AC and the sides of the unit squares it encounters are $A_1, A_2, \ldots, A_k$ in this order ($k \geq 2$), and $A_1 = A$ and $A_k = C$. Prove that

$$A_1A_2 - A_2A_3 + A_3A_4 - \cdots + (-1)^k A_{k-1}A_k = \frac{\sqrt{m^2 + n^2}}{mn}.$$